

THE SUSPENSION BRIDGE
WITH A STIFFENING TRUSS OF
VARIABLE MOMENT OF INERTIA

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WITH A STIFFENING TRUSS OF
VARIABLE MOMENT OF INERTIA

By

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SYNOPSIS

In the usual theory of suspension bridges, the moment of inertia of the stiffening truss is assumed to be constant and equal to the average value. This paper presents a study of the influence of variations in the moment of inertia of the truss on the horizontal component of the tension in the cable, the bending moments and the deflections of the stiffening truss. The derivation of the working formulas for bending moments and deflections, when a variable moment of inertia is involved, is easily accomplished thru the use of a trigonometric series.

Since the curve of the moment of inertia of the truss does not usually follow a simple algebraic form, it is necessary to assume an expression for the moment of inertia. Two assumptions are studied in this paper as well as the usual assumption of constant moment of inertia. In the first of these assumptions, the moment of inertia is represented by three horizontal lines. This is the same assumption as that made by Turneaure. In the second case, the moment of inertia is represented by two inclined lines and one horizontal line. This case approximates more nearly the true curve of the moment of inertia than does the previous case. The formulas derived from each assumption are similar and have the same degree of workability.

In the analysis of long-span suspension bridges, it was found that the horizontal component of the tension in the cable is independent of the moment of inertia for all practical purposes. For bridges with a span of 1500', the variable moment of inertia has small effect on the bending moments and deflections. There is possibility for a greater effect though in longer span bridges and heavier live loads. The second assumption of moment of inertia gives a simpler and more exact solution than the method of Turneaure.



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INTRODUCTION

Suspension bridges can be divided into two classes. The first class is called Unstiffened Suspension Bridges. In this group are found bridges which have a light weight pathway or roadway suspended from the cable and the loads are transmitted directly to the cable. When heavy or unsymmetrical loads are applied to a structure of this type, the distortion that takes place causes an excessive change of grade in the region of the load and so the bridge is not suitable for heavy loads. The other class is called Stiffened Suspension Bridges. In this class, there is a truss which tends to distribute the applied load thru the hangers to the cable as a uniform load. The action of the truss therefore reduces the deflection at the point of application of the load and the bridge is then able to carry heavier loads than without the truss.

The stiffened suspension bridges can be divided into two groups. The first group consists of those bridges whose cables have been made rigid to resist distortion by the use of a truss system built along the cable. This class is usually referred to as the Braced Chain or Braced Cable Bridge. The analysis of a structure of this type is that for an inverted braced arch. In the other class, there is a horizontal stiffening truss suspended from the cable. The truss may be entirely supported by the cables in the main span with the side spans carried on bents. The main span truss may be hinged at the towers or it may be continuous thru the towers. The type of structure to be studied in this paper is a suspension bridge over three openings with the horizontal stiffening truss hinged at the towers and the backstays loaded.

The two theories which are usually used in the analysis of suspension bridges with horizontal stiffening truss are the Elastic or Approximate Theory and the Deflection or Exact Theory. In the Elastic Theory the stiffening trusses are considered to be very stiff so that deflections are small. With this restriction, the effect of deflections on bending moments is neglected. The cable then retains its initial form as far as analysis is concerned, and the hanger loads are uniformly distributed under all types of loading. This is only a very approximate solution as the bending moments are in error as much as 70% for spans of 1000 feet and over. The error, however, is on the side of safety.

The Exact Theory developed by Melan¹ and reviewed and

¹Melan - Eiserne Bogenbrücken und Hängebrücken 1888.

amplified by others,^{2,3,4} takes into account the effect of deflection under load. The effect of using the deflection in the analysis is to reduce the bending moment as given by the approximate theory and thereby bring about a considerable saving in material. This reduction in bending moment for short span bridges is not as great as for long span bridges but the exact theory should be used for all structures except very short spans so as to bring about the greatest economy possible. The Deflection Theory is presented in this paper for the analysis of the long span suspension bridge.

In the analysis to determine the stresses in a suspension bridge with a horizontal stiffening truss, some assumptions are made. They are:

1. The initial curve of the cable is a parabola. If the dead weight of the structure were all distributed uniformly along the horizontal, the curve of the cable would be a parabola. In long span bridges, as the Ambassador, the Manhattan, and the Philadelphia-Camden, about 80% of the total dead weight is distributed uniformly along the horizontal and the other 20% is distributed along the cable. So the assumption that all the dead weight is distributed uniformly along the horizontal is a reasonable one and therefore the initial curve of the cable is a parabola.

2. The dead load stresses in the stiffening truss are zero and the entire dead load is carried by the cable. This is made possible thru proper procedure in erection. The truss is erected in the usual way but the joints are not riveted until the entire truss has been brought into position by adjustment of the hangers with allowance made for change of temperature from normal.

3. The elongation of the hangers under live load or temperature is neglected because it is a quantity very small in comparison with other terms. So the deflection of the truss is considered equal to that of the cable.

4. The horizontal component of the tension of the cable in the side span is equal to that of the main span. This is true if the cable is fixed at the top of flexible towers or if the cable is fixed to a movable saddle at the top of the towers.

5. The spacing of the hangers is so small in comparison with the span length that the hangers may be considered as forming a continuous sheet.

²D. B. Steinmann - A Practical Treatise on Suspension Bridges 1929.

³Johnson, Bryan and Turneure - Modern Framed Structures 9th Ed., Vol. II, 1910.

⁴L. S. Moisseiff - The Delaware River Bridge - Final Report of Board of Engineers, R. Modjeski, Chairman.

6. The stiffening truss is initially straight and horizontal and carries no dead load as is evident from the second assumption.

7. The live load is distributed uniformly to the cables. When the horizontal component of the tension in the cable is calculated using the actual distribution of the live load to the cables, it is only 1% different than when the actual distribution is neglected."

In the analysis of long-span, stiffened suspension bridges the moment of inertia is assumed as constant and equal to the average value. In practice, the moment of inertia is not constant as may be seen in Fig. 1 which shows the variation of the moment of inertia over one-half of the main span for each of the following bridges: Ambassador Bridge at Detroit, the Manhattan Bridge at New York, and the Philadelphia-Camden Bridge. The Ambassador Bridge is for highway traffic only, while the other two bridges also carry rapid-transit systems. These variations in the moment of inertia can be taken as typical of that present in all long-span bridges.

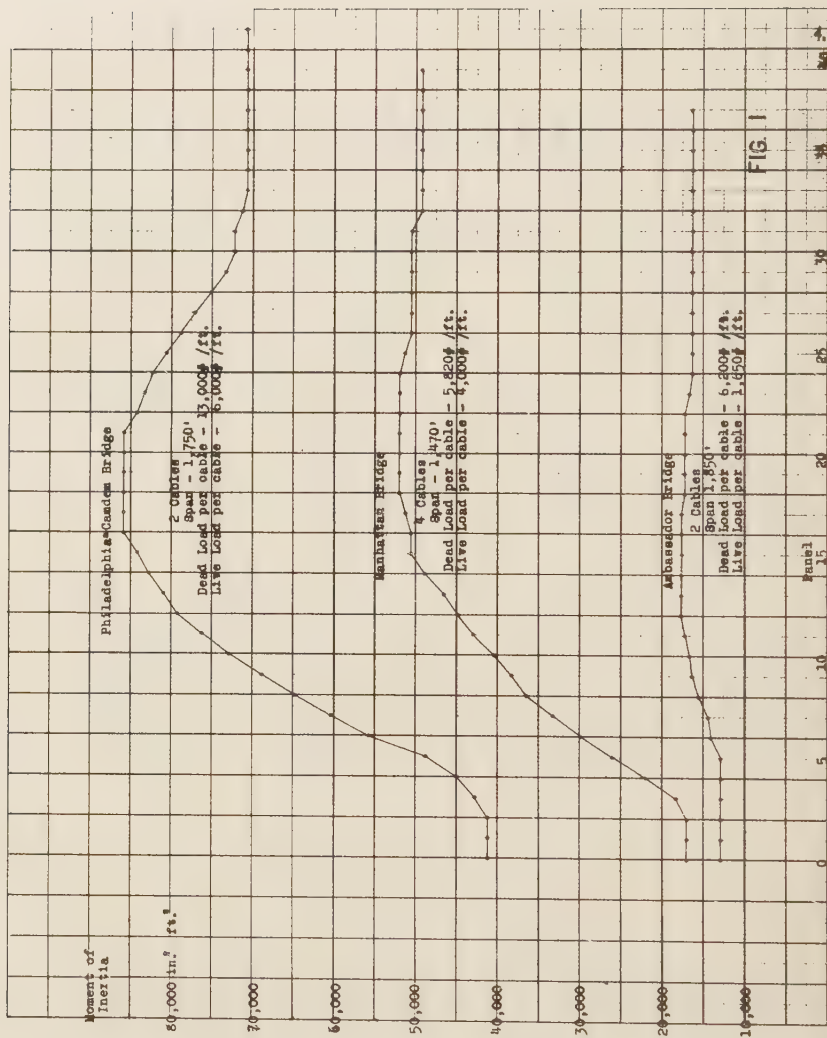
The purpose of this dissertation is to show what error there is in the assumption that the moment of inertia is constant and to present two methods for taking the variation into account that are quicker and simpler than those suggested in the literature on the subject. The literature is not extensive, being represented by the work of F. E. Turneaure,⁵ G. B. Martin⁶, and E. Steurmann⁷. These methods will be reviewed, and the last two are presented in the appendix. Two new methods are developed and are compared with the method of Turneaure by means of a numerical problem.

⁵V. A. Roudoy - The Analysis of Suspension Bridges by the Use of Fourier Series. - A thesis presented for degree of Master of Science in Engineering. U. of Mich. 1934.

⁶Ibid pg. 302

⁶G. B. Martin - Engineering, Vol. 125, pg. 1.

⁷E. Steurmann - Transactions of American Society of Civil Engineers, Vol. 94, page 394.



NOMENCLATURE

Nomenclature, not evident from Fig. 2, is as follows:

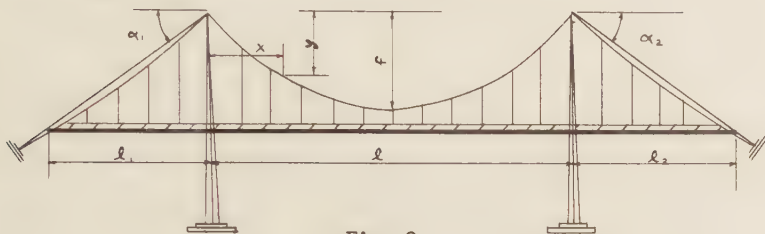


Fig. 2

w = dead load per unit length.

p = live load per unit length.

q = portion of live load carried by cable.

H_w = horizontal component of cable stress produced by dead load.

H_p = horizontal component of cable stress produced by live load.

H_s = horizontal component of cable stress produced by all causes except dead load.

$H = H_w + H_s$.

η = deflection of cable = deflection of stiffening truss.

E = modulus of elasticity of material in stiffening truss.

E_c = modulus of elasticity of material in cable.

A_c = area of cable.

M = bending moment in truss.

M_1 = bending moment in truss due to live load only assuming the truss to be a simply supported beam.

I = moment of inertia at any point in the truss.

I_A = average moment of inertia of truss.

$$\beta = \frac{H_s}{H_w}$$

$y = \frac{4fx}{l} (l - x)$ equation of cable under dead load.

THE VARIATION OF H_p, η , AND M WITH I_A

It is of interest to note how the values of the horizontal component of the cable stress, deflections, and bending moments change with arbitrarily assumed variations of the moment of inertia. The type of bridge to be used in this example, is a suspension bridge with a two hinged stiffening truss and unloaded backstays.

The deflection theory was presented originally by Melan and has been reviewed and enlarged upon by others. In this theory, the moment of inertia is taken as constant throughout the truss and equal to the average value. A brief summary of the theory and the necessary equations are presented here.

In Fig. 3, the bending moment at any point A as given by the deflection theory is:

$$M = M_1 - (H_w + H_s)\eta - H_s y \quad (1)$$

The deflection and bending moment are related by the following equation:

$$EI \frac{d^2 \eta}{dx^2} = -M \quad (2)$$

The solution of the differential equation for η brings about the following form:

$$\eta = C_1 e^{ax} + C_2 e^{-ax} + \frac{1}{H} \left(M_1 - \frac{P}{a^2} \right) - \frac{H_s}{H} \left(y - \frac{8f}{a^2 l^2} \right) \quad (3)$$

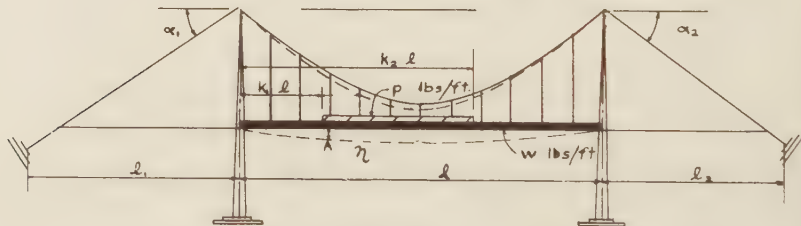


Fig. 3

where C_1 and C_2 are the constants resulting from the integration and $a^2 = \frac{H_w + H_s}{EI}$. Provided H_s is known, the constants of integration can be determined from the end conditions and conditions of continuity.

The equation for H_s may be developed by the energy method of Melan assuming that the hanger loading under live load is

constant, or by the geometric method of Bleich⁸ and Krivoshein⁹.

$$\frac{H_s}{A_c E_c} L \pm \omega t L_t = \frac{8f}{\ell^2} \int_0^{\ell} \eta \, dx \quad (4)$$

$$\text{Where } L = \int_0^{\ell} \frac{ds^3}{dx^3} + \ell_1 \sec^3 \alpha_1 + \ell_2 \sec^3 \alpha_2$$

$$\text{and } L_t = \int_0^{\ell} \frac{ds^2}{dx} + \ell_1 \sec^2 \alpha_1 + \ell_2 \sec^2 \alpha_2$$

The evaluation of eq.(4) involves the integration of the quantity ηdx . In eq.(3), the algebraic form of η changes as M_1 changes from one section to another. For the general loading indicated in Fig. 3, there will be three forms for the quantity η and six constants of integration which can be determined from the boundary conditions and from equations of continuity. With the algebraic form of these constants, the values of η from eq.(3) can be substituted in eq.(4) with the proper limits of integration and H_s thereby determined.

The working formula for H_s as presented by A. A. Jakkula¹⁰ is:

$$\begin{aligned} \frac{L}{A_c E_c} H_s^2 + H_s \left[\frac{H_w L}{A_c E_c} \pm \omega t L_t + \frac{16f^2}{3\ell} - \frac{64f^2 K_2}{\ell^4} \right] \\ + \frac{pf\ell K_1}{3} \pm \omega t L_t H_w + \frac{8fpK_3}{\ell^2} = 0 \end{aligned} \quad (5)$$

Where K_1 , K_2 and K_3 have the following values,

$$K_1 = k_2^2 (4k_2 - 6) - k_1^2 (4k_1 - 6) \quad (6)$$

$$K_2 = \frac{4}{a^3 (e^{a\ell} - e^{-a\ell})} + \frac{a\ell - 2}{a^3} \quad (7)$$

$$K_3 = \left(\frac{e^{-ak_2\ell} - e^{-ak_1\ell}}{a^3 (e^{a\ell} - 1)} \right) + \left(\frac{e^{ak_2\ell} - e^{ak_1\ell}}{a^3 (e^{a\ell} - 1)} \right) + \frac{ak_2\ell - ak_1\ell}{a^3} \quad (8)$$

A direct solution of eq.(5) is not possible for the quantities K_2 and K_3 involve "a" which is a function of H_s . From the nature of eq.(5), it was found that K_2 and K_3 would have to be of considerable value to affect the other terms.

⁸F. Bleich - Theorie und Berechnung der Eisernen Brucken pg. 459.

⁹G. G. Krivoshein - Simplified Calculation of Statically Indeterminate Bridges - pg. 280.

¹⁰A. A. Jakkula - The Theory of the Suspension Bridge - A dissertation presented for the degree of Doctor of Philosophy in Engineering. U. of Mich., 1933.

"F. E. Coe - The Application of Differential Equation Method Formulas to the Ambassador Bridge - Thesis presented for the degree of Master of Science in Engineering. U. of Mich., 1933.

So, for the first approximation, K_2 and K_3 are assumed equal to zero. The quadratic equation is then easily solved for H_2 . Then, with this value of H_2 , the quantities K_2 and K_3 are computed and a new solution of eq.(5) is made. For most cases of loading investigated by this writer¹², the difference between the first and second approximations was less than one per cent. When a third solution was tried, it was found that the second and third values were identical for all practical purposes.

It was also noticed that the term which involves the moment of inertia appears in eq.(5) in the quantities K_2 and K_3 only. So it was evident that the value of the moment of inertia has little or no effect on H_2 . A study is presented here of the effect on H_2 of arbitrarily changing the moment of inertia from twice the average value to one-half the average value using the Ambassador Bridge for an example.

The necessary dimensions and quantities for the Ambassador Bridge¹² are as follows:

$l = 1850'$	$f = 205.6'$
$l_1 = 989.4'$	$E_c = 27,000,000\#/sq.in.$
$l_2 = 833.9'$	$E = 30,000,000\#/sq.in.$
$\alpha_1 = 20^\circ 32'$	$w = 6,200\#/ft./cable$
$\alpha_2 = 24^\circ 00'$	$p = 1,000\#/ft./cable$ (assumed)
$I_A = 113,71'$ (calculated) ¹³	$H_w = 12,920,000\#$
$L = 4338.09$	$A_c = 240.89$ sq.in.

The affect of temperature will be neglected in this part of the discussion. H_p is found by the use of eq.(5) with K_2 and K_3 equal to zero. With this value of H_p , and $I = I_A$, the term "a" is calculated by use of the relationship $a^2 = \frac{H_w + H_p}{EI}$. The second computation is made using the terms $\frac{H_w}{EI}$ involving K_2 and K_3 . Using this second determination of H_p , trial values of "a" are computed for values of I varying from $2I_A$ to $\frac{I_A}{2}$. The resultant values of H_p for live load on the end quarter of the span are shown in Table I. For end loading, the percentage difference between the value of H_p as found for I_A and as found for $2I_A$ was .73%. For end quarter loading this percentage amounted to 1.48%. Fig. 4 shows the results graphically for the same two loading conditions. Since the change in H_p is so small, the value of H_p as found using $I = I_A$ will be used in determining the deflection and bending moment for a change in I from $2I_A$ to $\frac{I_A}{2}$. The deflection is found from eq.(3) and the bending moment from eq.(1). Table I shows the deflection and bending moment $x = .1l$ for live load on the end half of the span. The deflec-

¹²Report of McClintic-Marshall Company, 1930.

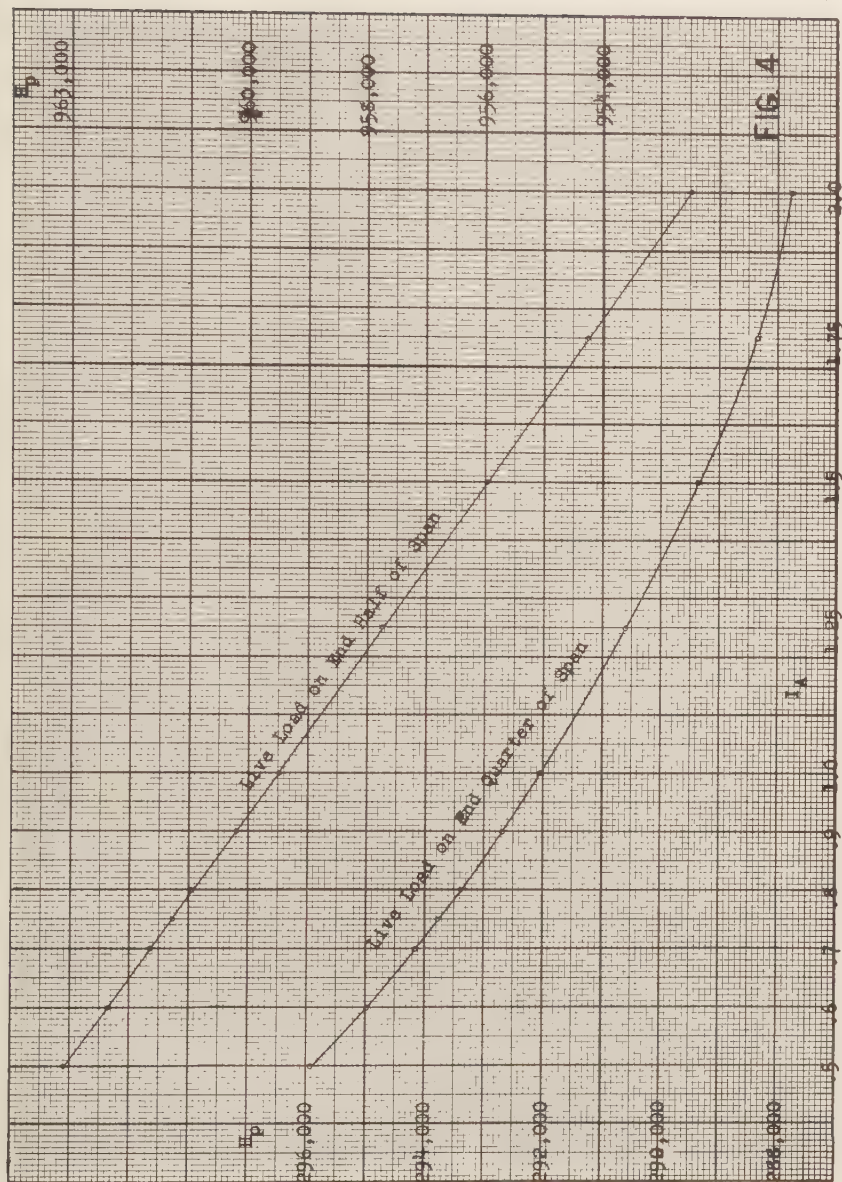
¹³The moment of inertia is computed using chord members only.

tion when $I = 2I_A$ is 19% less than when $I = I_A$ and the change in bending moment is 51%. The deflection and moment curves are plotted in Fig. 5. It is seen that there is considerable curvature in both quantities. From the standpoint of design, it is desirable to know the effect of change of the moment of inertia over a short range. So the procedure was continued from $.9I_A$ to $1.1I_A$ with increments of one per cent of I_A . Table II shows the results for η and M at $x = .3l$ for end half loading and at $x = .2l$ for end quarter loading. Fig. 6 shows the results in a graphical form for end half loading. The curves are seen to very closely approximate straight lines within a range of 10% change in I .

The fact that H_p may be considered independent of the moment of inertia is very important in the procedures to be developed. Although the affect on H_p is small, the affect on η and M is considerable, so that the variation in the moment of inertia should be considered.

TABLE I

I _A	End Half Loading		End Quarter Loading		End Half Loading		$\Sigma = .1\%$	
	H _p -lbs.	diff.	H _p -lbs.	diff.	γ ft.	diff.	M ft.lbs.	diff.
2.00	952,490	.73%	287,730	1.48%	1.6647	19%	17,638,400	51%
1.75	954,220	.55%	288,290	1.29%	1.7450	15%	16,393,300	41%
1.50	955,960	.37%	289,370	.91%	1.8357	11%	15,002,500	29%
1.25	957,720	.18%	290,600	.49%	1.9391	6%	13,434,100	15%
1.00	959,490		292,040		2.0585		11,642,800	
.90	960,200	.07%	292,690	.22%	2.1117	3%	10,850,600	7%
.80	960,920	.15%	293,390	.46%	2.1684	5%	10,000,200	14%
.75	961,280	.19%	293,770	.59%	2.1983	7%	9,566,300	18%
.70	961,640	.22%	294,160	.72%	2.2293	8%	9,108,800	22%
.60	962,370	.30%	294,990	1.02%	2.2947	11%	8,145,100	30%
.50	963,100	.38%	295,910	1.32%	2.3655	15%	7,106,400	39%



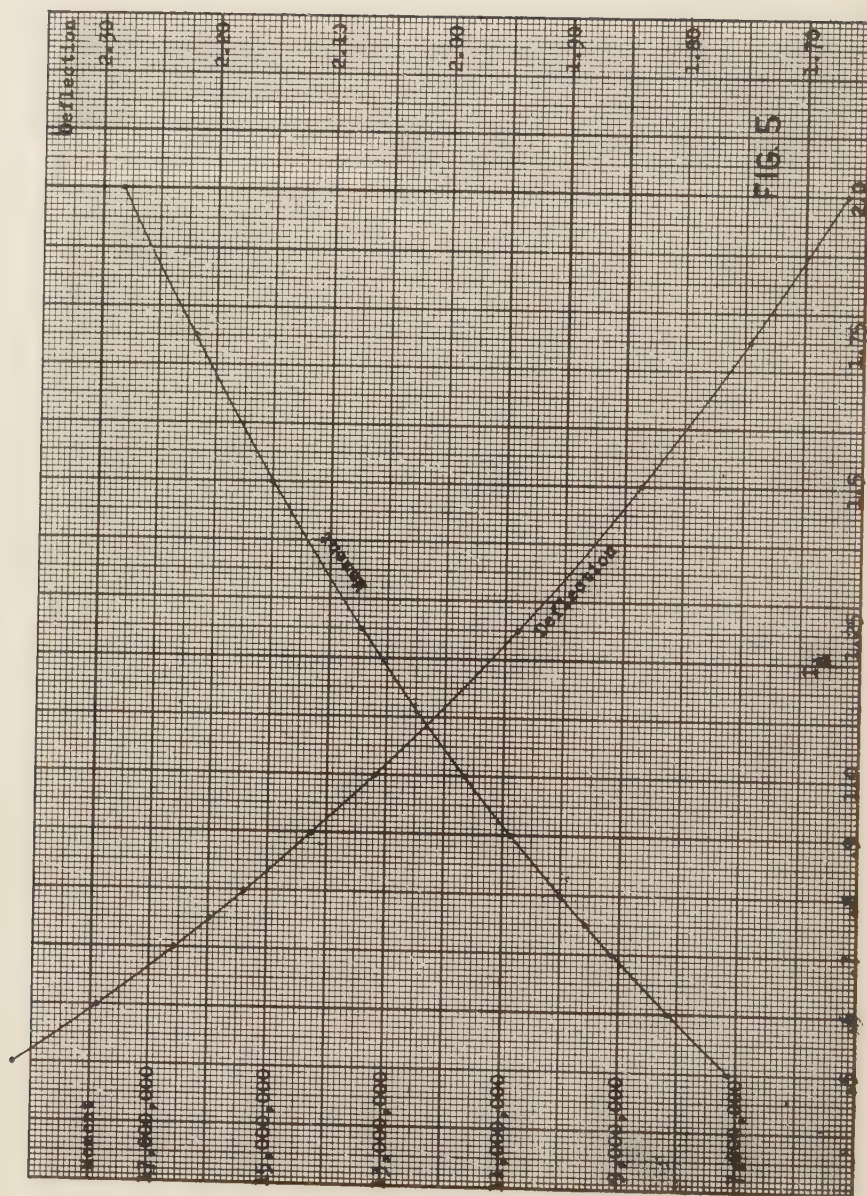


TABLE II

	End Half $x = .3\%$		End Quarter $x = .2\%$	
	η	M	η	M
.90 I _A	3.6673	14,410,200	2.1049	15,046,800
.91	3.6584	14,534,300	2.0972	15,147,800
.92	3.6495	14,657,800	2.0897	15,247,500
.93	3.6407	14,780,400	2.0823	15,345,400
.94	3.6319	14,902,200	2.0748	15,443,800
.95	3.6231	15,024,100	2.0674	15,542,200
.96	3.6144	15,144,800	2.0602	15,636,700
.97	3.6058	15,265,000	2.0530	15,732,600
.98	3.5971	15,384,600	2.0459	15,826,400
.99	3.5885	15,504,400	2.0386	15,921,900
1.00	3.5800	15,621,700	2.0316	16,014,600
1.01	3.5716	15,739,300	2.0246	16,107,800
1.02	3.5631	15,856,600	2.0176	16,199,500
1.03	3.5547	15,973,000	2.0107	16,290,600
1.04	3.5463	16,090,000	2.0039	16,381,000
1.05	3.5381	16,204,200	1.9971	16,471,200
1.06	3.5298	16,318,800	1.9904	16,559,600
1.07	3.5216	16,432,800	1.9837	16,647,600
1.08	3.5134	16,546,200	1.9770	16,736,200
1.09	3.5053	16,659,200	1.9704	16,823,600
1.10	3.4972	16,771,500	1.9638	16,910,500

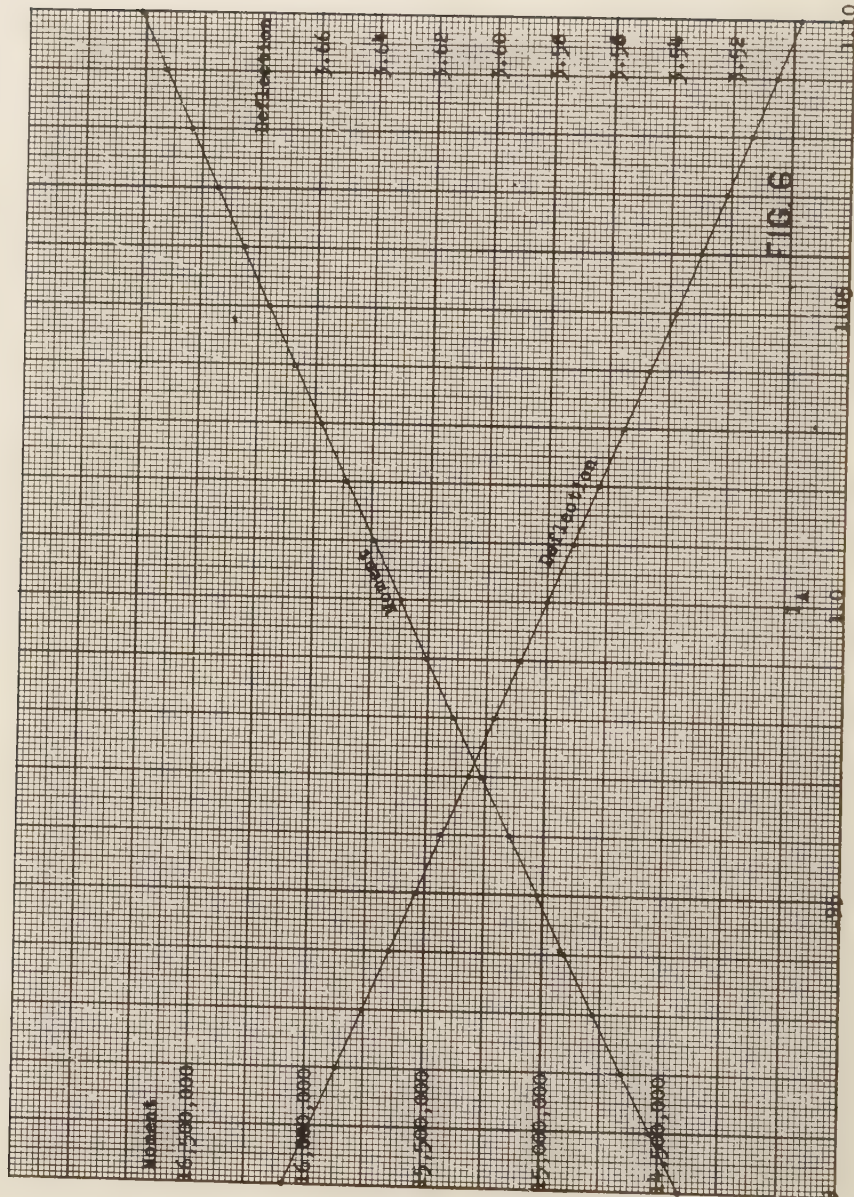


FIG. 6

DEFLECTION OF SIMPLE BEAM WITH VARIABLE MOMENT OF INERTIA

In Fig. 1 it is seen that the moment of inertia varies through a considerable range. Since the variation does not usually follow a curve that can be expressed in a simple algebraic form, the question naturally arises as to what simplifying approximations can be made in the moment of inertia. To gain some idea of how this approximation is to be made, a study was conducted of the effect on the deflection of a simple beam using some arbitrarily assumed variations of the moment of inertia.

For the purposes of this study, a beam will be assumed to be simply supported at the ends and loaded at the third points as shown in Fig. 7. The moment of inertia over the center half is assumed to be I_0 and at the ends of the beam it is I_1 . From each end of the beam to the respective quarter points, the moment of inertia varies as a straight line from I_0 to I_1 .

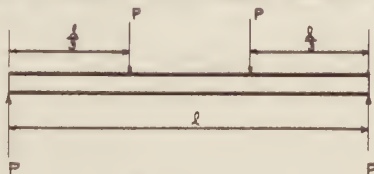


FIG. 7

The conjugate beam method¹ is used for determining the deflections. The $\frac{M}{EI}$ diagram is plotted and the bending moment found for each $\frac{M}{EI}$ point with the $\frac{M}{EI}$ diagram acting as the load. The solution will be made, (a) using the exact value of the moment of inertia $I = I_1 + (I_0 - I_1) \frac{4x}{l}$ when $0 < x < \frac{l}{4}$,

$I = I_0$ when $\frac{l}{4} < x < \frac{3l}{4}$, $I = I_1 + (I_0 - I_1) \frac{4(l-x)}{l}$ when $\frac{3l}{4} < x < l$, (b) using $I = I_0$ where I_0 is the average value of the moment of inertia throughout the beam, (c) using $I = I_1$ when $0 < x < \frac{l}{4}$ and $\frac{3l}{4} < x < l$ where I_1 is the average value over the end quarter and $\frac{l}{4} I = I_0$ when $\frac{l}{4} < x < \frac{3l}{4}$. The deflection is found for several points on the beam $\frac{l}{4}$ for each of the three cases. The numerical results in Table III are for $I_0 = 2I_1$ and for $I_0 = 3I_1$ and the numbers in the Table are coefficients of $\frac{Pl^3}{EI}$. It is seen that assumption in case (c) offers a very great improvement over the assumption in case (b). Also judging from the results of this numerical problem, there is reason for attempting the solution of the suspension bridge under case (a). That is to say, the assumed curve for the moment of inertia would follow that indicated in case (a) and this curve approximates the moment of inertia curve of a suspension bridge more closely than do either of the assumptions.

¹Seeley: Resistance of Materials, pg.164.

TABLE III

Case x	$I_0 = 2I_1$					$I_0 = 3I_1$				
	a	b	c	$\frac{b-a}{a}\%$	$\frac{c-a}{a}\%$	a	b	c	$\frac{b-a}{a}\%$	$\frac{c-a}{a}\%$
.1	.011524	.012508	.011930	8.5	3.5	.011848	.014593	.012424	23.2	4.9
.2	.021666	.023873	.022528	10.2	4.0	.022106	.027852	.023167	26.0	4.8
.25	.026089	.028770	.026909	10.3	3.1	.026405	.033565	.027777	27.1	5.2
.3	.029799	.032952	.030569	10.6	2.6	.030065	.038444	.031437	27.9	4.6
.333	.031863	.035273	.032600	10.7	2.3	.032096	.041152	.033468	28.2	4.3
.4	.034893	.038660	.035563	10.8	1.9	.035058	.045103	.036431	28.6	3.9
.5	.036659	.040564	.037229	10.6	1.6	.036725	.047325	.038097	28.9	3.7
Av.				10.2	2.7				27.1	4.5

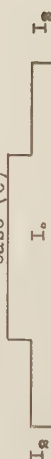
Case (a)



Case (b)



Case (c)



METHOD OF F. E. TURNEAURE³

Since it has been shown that the horizontal component of the tension in the cable is, for all practical purposes, independent of the moment of inertia, the problem resolves itself into finding a more exact solution for the deflection and bending moment by taking into account the variation of the moment of inertia.

Turneure divided the moment of inertia curve into three sections and found the average value of the moment of inertia from each end to the quarter point and then determined the average value for the center half. From the appearance of the curves of Fig. 1, this is a reasonable assumption. The study of the simple beam indicates a very reasonable accuracy may be expected. Had the curves been divided into five or more sections the work of the solution would have been very greatly increased for only a slight gain in accuracy.

Another equation for bending moment is found by taking the second derivative of η , eq.(3), and multiplying it by $(-EI)$.

$$M = -H(c_1 e^{ax} + c_2 e^{-ax}) + \frac{P}{a^2} - \frac{8H_0 f}{a^2 l^2} \quad (9)$$

This equation holds only for sections where I and p are constant. Should either I or p change, then the constants of integration c_1 and c_2 will change. Before the bending moment can be solved for, the constants of integration must be found.

A general loading condition will be assumed as indicated in Fig. 8 with the live load extending over a portion of the beam from $k_1 l$ to $k_2 l$. The moment of inertia will be called I_0 for a distance $j l$ from each end and I_c over the central portion of the span.

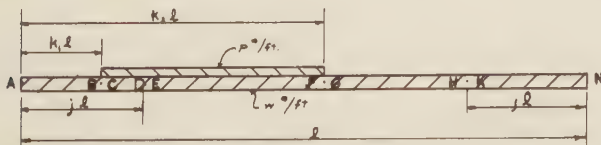


FIG. 8

Under the general conditions assumed in Fig. 8 there will be five moment equations, one for each section AB, CD, EF, GH, and KN. Thus there are ten constants of integration to be found. These may be obtained by placing the moments and shears at the end of one section equal to those at the beginning of the next, noting of course that M_A and M_N are equal to zero. In the following equations, $a_2^2 = \frac{H}{EI}$, $a_3^2 = \frac{H}{EI}$, and C_1 to C_{10} inclusive are the constants of integration. Eq.(9) has been transformed slightly to make it more workable in the following equations:

$$C_1 + C_2 + \frac{1}{a_2^2} \left(\frac{8f}{l^2} \right) = 0 \quad (10)$$

$$C_1 e^{a_1 x} + C_2 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} \right) = C_3 e^{a_1 x} + C_4 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} - \frac{p}{H} \right) \quad (11)$$

$$C_3 e^{a_1 x} + C_4 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} - \frac{p}{H} \right) = C_5 e^{a_1 x} + C_6 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} - \frac{p}{H} \right) \quad (12)$$

$$C_5 e^{a_1 x} + C_6 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} - \frac{p}{H} \right) = C_7 e^{a_1 x} + C_8 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} \right) \quad (13)$$

$$C_7 e^{a_1 x} + C_8 e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} \right) = C_9 e^{a_1 x} + C_{10} e^{-a_1 x} + \frac{1}{a_2^2} \left(\frac{8f}{l^2} \right) \quad (14)$$

$$C_9 e^{a_1 l} + C_{10} e^{-a_1 l} + \frac{1}{a_2^2} \frac{8f}{l^2} = 0 \quad (15)$$

$$C_1 e^{a_1 x} - C_2 e^{-a_1 x} = C_3 e^{a_1 x} - C_4 e^{-a_1 x} \quad (16)$$

$$a_2 (C_3 e^{a_1 x} - C_4 e^{-a_1 x}) = a_0 (C_5 e^{a_1 x} - C_6 e^{-a_1 x}) \quad (17)$$

$$C_5 e^{a_1 x} - C_6 e^{-a_1 x} = C_7 e^{a_1 x} - C_8 e^{-a_1 x} \quad (18)$$

$$a_0 (C_7 e^{a_1 x} - C_8 e^{-a_1 x}) = a_2 (C_9 e^{a_1 x} - C_{10} e^{-a_1 x}) \quad (19)$$

The general solution of these ten simultaneous equations would involve much work. Turneaure presents the solution of these equations in a numerical problem, but not in general terms. Under certain conditions of loading the number of constants may be reduced to eight or six. But the solution of six, eight or ten simultaneous equations requires much care. After the constants are found, the bending moment may be determined in the usual way. The solution for moment under the assumptions made will be numerically exact. A numerical example will be introduced later for a comparison of this method with the proposed methods.

PROPOSED METHODS

The two proposed methods will differ only in the assumed form of the curve for the moment of inertia. The first form will be similar to that for case (c) for a simple beam considered previously in this paper, while the second form is that of case (a) for a simple beam. The derivation of formulas will be carried through for the first case and as the proofs are identical, the results only will be given for the second case.

The series method for the elastic curve of the stiffening truss under live load has been presented by S. Timoshenko¹⁵, G. Priestler¹⁶, and A. A. Jakkula¹⁰. In these developments, the moment of inertia was considered as constant from tower to tower and equal to the average value. As has already been shown in this paper, H_1 is, for all practical purposes independent of the moment of inertia. Therefore, it is necessary to formulate equations for deflection and bending moment.

Case A - The moment of inertia is taken as constant and equal to the average value over the distance of $j\ell$ from each end of the truss and this quantity is called I_2 . The average value of the moment of inertia over the central portion is called I_1 . The dead load of $w\#/ft.$ extends across the entire span and the live load of $p\#/ft.$ extends from $k_1\ell$ to $k_2\ell$. The hanger pull is equal to $(w+q)\#/ft.$ where q is the amount of live load carried by the hangers. Fig. 9 illustrates this loading condition.

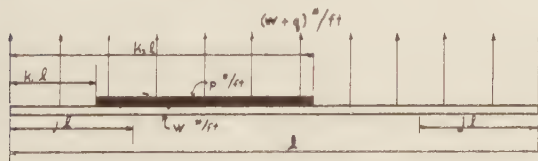


FIG. 9

The deflection curve can be expressed as trigonometric series which reduces to a sine series because of the end conditions.

$$\gamma = a_1 \sin \frac{\pi x}{\ell} + a_2 \sin \frac{2\pi x}{\ell} + \dots + a_n \sin \frac{n\pi x}{\ell} \quad (20)$$

¹⁵S. Timoshenko - Transactions of American Society of Civil Engineers, Vol. 94, pg. 377.

¹⁶George Priestler - Engineering Research Bulletin No. 12, U. of Mich.

¹⁰ibid.

The coefficients a_1, a_2 , etc., must be calculated before the deflection can be found. A method of making this evaluation is to set up an equation for the potential energy of the system. Let U = potential energy.

$U = \int_0^l \frac{M^2}{2EI} dx$ (21) is the equation for the potential energy stored in a beam. Remembering that $M = -EI \frac{d^2 \eta}{dx^2}$, the equation for potential energy becomes

$$U = \frac{E}{2} \int_0^l I \left(\frac{d^2 \eta}{dx^2} \right)^2 dx \quad (22)$$

Since the moment of inertia is not constant over the entire truss, the factor I must be placed under the integral sign. However, under the assumption made, the moment of inertia is constant within certain definite limits, so eq.(22) now takes the form,

$$U = \frac{E}{2} \left\{ \int_0^l I_2 \left(\frac{d^2 \eta}{dx^2} \right)^2 dx + \int_l^{l-j} I_0 \left(\frac{d^2 \eta}{dx^2} \right)^2 dx + \int_{l-j}^l I_2 \left(\frac{d^2 \eta}{dx^2} \right)^2 dx \right\} \quad (23)$$

From Eq.(20)

$$\frac{d^2 \eta}{dx^2} = - \left[\frac{\pi^2}{l^2} \sin \frac{\pi x}{l} + \frac{4\pi^2}{l^2} \sin \frac{2\pi x}{l} + - - \frac{n^2 \pi^2}{l^2} \sin \frac{n\pi x}{l} \right]$$

When this equation is squared, there will be product terms of the form $2 \left(\frac{m\pi}{l} \right)^2 \left(\frac{n\pi}{l} \right)^2 a_m a_n \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}$ and there will be square terms as $a_n^2 \left(\frac{n\pi}{l} \right)^2 \sin^2 \frac{n\pi x}{l}$, where m and n are any two positive integers.

Then

$$\begin{aligned} U = \frac{E}{2} & \left\{ I_2 \int_0^l 2m^2 n^2 \left(\frac{\pi}{l} \right)^4 a_m a_n \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} + a_n^2 \left(\frac{n\pi}{l} \right)^4 \sin^2 \frac{n\pi x}{l} dx \right. \\ & + I_0 \int_l^{l-j} 2m^2 n^2 \left(\frac{\pi}{l} \right)^4 a_m a_n \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} + a_n^2 \left(\frac{n\pi}{l} \right)^4 \sin^2 \frac{n\pi x}{l} dx \\ & \left. + I_2 \int_{l-j}^l 2m^2 n^2 \left(\frac{\pi}{l} \right)^4 a_m a_n \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} + a_n^2 \left(\frac{n\pi}{l} \right)^4 \sin^2 \frac{n\pi x}{l} dx \right\} \quad (24) \end{aligned}$$

When integrated between the indicated limits and simplified, the equation for potential energy takes the form,

$$\begin{aligned} U = E \sum_{n=1}^{\infty} \frac{\pi^3}{2l^3} m^2 n^2 a_m a_n (1 + \cos m\pi \cos n\pi) \frac{\sin(m-n)\pi j}{(m-n)} \frac{\sin(m+n)\pi j}{(m+n)} (I_2 - I_0) \\ + \frac{1}{2} a_n^2 \left(\frac{n\pi}{l} \right)^3 \left[n\pi j - \sin n\pi j \cos n\pi j \right] (I_2 - I_0) + \frac{I}{4} a_n^2 \left(\frac{n\pi}{l} \right)^4 \left\{ \right. \quad (25) \end{aligned}$$

By hypothesis, $m \neq n$. From the term $(1 + \cos m\pi \cos n\pi)$, it is evident m and n must both be odd or both be even otherwise the first term of eq. (25) will become zero. If $I_0 = I_2$, the first two terms become zero and the equation assumes the form for a truss with constant moment of inertia as

$$U = \frac{EI_0}{4} \frac{\pi^4}{l^3} \sum_{n=1}^{\infty} a_1^2 + 2 a_2^2 + 3 a_3^2 + \dots + n^4 a_n^2$$

If a_n is increased by an amount Δa_n , then U will be increased by $\Delta a_n \sin \frac{n\pi x}{l}$ and the increase in potential energy will be

$$\begin{aligned} \Delta U = \frac{\partial U}{\partial a_n} \Delta a_n &= E \sum_{n=1}^{\infty} \frac{\pi^3}{l^3} m^2 n^2 a_n \left[\sin \frac{(m-n)\pi l}{(m-n)} - \sin \frac{(m+n)\pi l}{(m+n)} \right] (I_2 - I_0) \\ &+ a_n \left(\frac{n\pi}{l} \right)^3 \left[n\pi l - \frac{\sin 2n\pi l}{2} \right] (I_2 - I_0) + \frac{I}{2} a_n \left(\frac{n\pi}{l} \right)^4 \Delta a_n \end{aligned} \quad (26)$$

The external work is done by the loads and forces acting as shown in Fig. 9. The dead load w is balanced by the equal and opposite dead hanger forces, so there remains only to consider the live load p and the live load hanger forces q . The portion of the live load carried by the truss is $-\frac{d^2 M}{dx^2}$.

$$M = M_1 - H\eta - H_s y \quad (27)$$

$$\begin{aligned} -\frac{d^2 M}{dx^2} &= -\frac{d^2 M_1}{dx^2} + H \frac{d^2 \eta}{dx^2} + H_s y'' \\ &= p + H \frac{d^2 \eta}{dx^2} - H_s \frac{8f}{l^2} \end{aligned}$$

$$q = -H \frac{d^2 \eta}{dx^2} + H_s \frac{8f}{l^2} \quad (28)$$

$$q = -H_w(1 + \beta) \frac{d\eta}{dx} + \beta w \quad (29)$$

The forces p and q also act through the distance $\Delta a_n \sin \frac{n\pi x}{l}$, so the increase in external work becomes,

$$\Delta W_e = \Delta a_n p \int_{k_1 l}^{k_2 l} \sin \frac{n\pi x}{l} dx - \Delta a_n \int_0^l (\beta w - H_w(1 + \beta) \frac{d^2 \eta}{dx^2}) \sin \frac{n\pi x}{l} dx \quad (30)$$

$$\Delta W_e = \Delta a_n \left[\frac{pl}{n\pi} (\cos n\pi k_1 - \cos n\pi k_2) \right] - \Delta a_n \left[\frac{\beta w l}{n\pi} (1 - \cos n\pi) + H_w \frac{n^2 \pi^2}{2l} \right] \quad (31)$$

When the increase in potential energy is equated to the increase in external work, a general expression for a_n is obtained but it involves the term a_m .

$$a_n \left[\left(\frac{n\pi}{l} \right)^3 (n\pi j - \frac{\sin 2 n\pi j}{2}) (I_2 - I_0) E + \frac{EI_0}{2} \left(\frac{n\pi}{l} \right)^4 + \frac{H(n\pi)^2}{2l} \right] \\ + \sum a_m \left(\frac{m\pi}{l} \right)^3 m^2 n^2 \left[\frac{\sin(m-n)\pi j}{(m-n)} - \frac{\sin(m+n)\pi j}{(m+n)} \right] (I_2 - I_0) E = \\ \frac{pl}{n\pi} (\cos n\pi k_1 - \cos n\pi k_2) - \frac{\beta w l}{n\pi} (1 - \cos n\pi) \quad (32)$$

$$a_n = \frac{\frac{l}{n\pi} p (\cos n\pi k_1 - \cos n\pi k_2) - \beta w (1 - \cos n\pi)}{\left(\frac{n\pi}{l} \right)^3 (n\pi j - \frac{\sin 2 n\pi j}{2}) (I_2 - I_0) E + \frac{EI_0}{2} \left(\frac{n\pi}{l} \right)^4 + \frac{H(n\pi)^2}{2l}} \\ - \frac{\sum a_m \left(\frac{m\pi}{l} \right)^3 m^2 n^2 \left[\frac{\sin(m-n)\pi j}{(m-n)} - \frac{\sin(m+n)\pi j}{(m+n)} \right] (I_2 - I_0) E}{\left(\frac{n\pi}{l} \right)^3 (n\pi j - \frac{\sin 2 n\pi j}{2}) (I_2 - I_0) E + \frac{EI_0}{2} \left(\frac{n\pi}{l} \right)^4 + \frac{H(n\pi)^2}{2l}} \quad (33)$$

The procedure for the determination of particular values of a will be developed in the numerical problem. After the values of a_1 to a_n have been obtained, the deflection is found by using eq.(20) and the bending moment by eq.(19).

Case B - In the second case, the moment of inertia at each end of the truss is I_1 . From each end to a point distant $j_1 l$ from each end, the moment of inertia will vary as a straight line from I_1 at the end to I_0 at the point $j_1 l$ from the end. The moment of inertia over the central portion of the span is then similar to case (a) that was discussed under the simple beam. That is,

$$I = I_1 + (I_0 - I_1) \frac{x}{j_1 l} \quad 0 < x < j_1 l \\ I = I_0 \quad j_1 l < x < (l - j_1 l) \\ I = I_1 + (I_0 - I_1) \frac{l-x}{j_1 l} \quad (l - j_1 l) < x < l$$

The method of obtaining the general equation for a is exactly the same as in the first method, so the detailed solution will not be repeated, but only the final equation for a_n will be given.

$$\begin{aligned}
 a_n = & \frac{\left(\frac{l}{n\pi}\right)^3 E \left[p(\cos n\pi k_1 - \cos n\pi k_2) - \beta w (1 - \cos n\pi) \right]}{\frac{n^3 \pi^3}{2} \left[I_0 - (I_0 - I_1) \left(j - \frac{1 - \cos 2n\pi j}{2 n^2 \pi^2 j} \right) \right] + \frac{H l^3}{2 E}} \\
 & + \frac{\sum_{m=0}^{\infty} a_m m^2 \left(\frac{I_0 - I_1}{j} \right) \left[\frac{1 - \cos(m-n)\pi j}{(m-n)^2} - \frac{1 - \cos(m+n)\pi j}{(m+n)^2} \right]}{\frac{n^3 \pi^3}{2} \left[I_0 - (I_0 - I_1) \left(j - \frac{1 - \cos 2n\pi j}{2 n^2 \pi^2 j} \right) \right] + \frac{H l^3}{2 E}} \quad (34)
 \end{aligned}$$

As in the preceding case, the actual solution for a_n will be developed in the numerical problem. The deflections and bending moments are calculated in the usual way.

For comparison, the value of a_n , when the moment of inertia is assumed constant throughout the span and equal to I_0 , is introduced here,

$$a = \frac{\frac{2l}{n^3 \pi^3 E} \left[p(\cos n\pi k_1 - \cos n\pi k_2) - \beta w (1 - \cos n\pi) \right]}{n^2 \pi^2 I_0 + \frac{H l^3}{E}} \quad (35)$$

The first case presented here is quicker and easier to solve than the solution presented by Turneaure for the same assumption as to the moment of inertia curve. In the second case considered, the assumed inertia curve more closely approximates the moment of inertia curve than does that of the first method, therefore a more accurate solution is obtained. The work of the second case is no more than that of the first and for this reason the second case is to be preferred when making a solution.

ILLUSTRATIVE PROBLEM

The Manhattan Bridge at New York City is used as a numerical example for a comparison of the method of Turneaure and the proposed methods. This bridge has a two hinged stiffing truss in the main span and loaded backstays. The chief dimensions and data¹⁷ concerning the bridge are:

$l = 1446.7'$	$E = 29,000,000\#/sq."'$	$I_A = 43,900 \text{ in.}^2\text{ft.}^2$
$l_1 = l_2 = 713.5'$	$E_c = 27,000,000\#/sq."'$	$I_o = 47,150 \text{ in.}^2\text{ft.}^2$
$f = 145.3'$	$A_c = 275 \text{ sq."}$	$I_s = 35,100 \text{ in.}^2\text{ft.}^2$
$f_1 = f_2 = 37.2'$	$p = 4000\#/ft./\text{cable}$	$I_1 = 16,440 \text{ in.}^2\text{ft.}^2$
$L = 3,452$	$w = 5,820\#/ft./\text{cable}$	
$L_c = 3,324$	$H_w = 10,480,000\#$	

The type of loading studied is that which produces maximum positive moment in the main span. According to Turneaure, this loading is live load from 0 to .425 ℓ and the temperature 55° above normal. The value of H , as calculated by Turneaure is 2,022,000 lbs. The moment of inertia is computed on the basis of chord members with an increase of 8.2% made for the affect of web members.

Turneaure divided the inertia curve into three parts and found the average value from each end to the quarter points, that is, $I_s = 35,100 \text{ in.}^2\text{ft.}^2$, and then found the average value of the moment of inertia over the center half, that is $I_o = 47,150 \text{ in.}^2\text{ft.}^2$. This is seen to be a reasonable assumption from the appearance of the curve for the Manhattan Bridge in Fig. 1. A further subdivision of the moment of inertia curve would have greatly increased the work for a small gain in accuracy.

The general solution according to this method requires ten equations, but under this condition of loading the number of constants is reduced to eight, so there are eight equations only. Although all eight constants do not appear in each equation, the solution requires considerable time. The bending moment, after the constants are found, is computed by the use of eq.(9). The results are indicated in Table IV under the heading of eq.(9). For the sake of comparison the bending moments, as calculated using a constant moment of inertia across the entire beam equal to I_A , are placed in the column under C.M.I. For the maximum moment at the quarter point there is

¹⁷Ralph Modjeski, City of New York, Department of Bridges, Manhattan Bridge.

¹⁸ibid pg.297.

seen to be a reduction of 3.2% by using a variable moment of inertia.

For the Case A, the assumption as to moment of inertia is exactly the same as Turneaure's. The number of terms to be used in determining the deflection by the series method is set at six, as four did not give the desired accuracy, and five produced an unsymmetrical solution for the coefficient a_n . In eq.(25) it was seen that "m" and "n" both had to be even or both odd, so the general form for a_n in eq.(33) will degenerate into two parts, one for even values of m and n, and one for odd values. Eq.(32) is transformed for the solution into,

$$\begin{aligned} a_n \left[\frac{n^2 \pi^2}{2} (I_0 - (I_0 - I_2)(j - \frac{\sin 2 n \pi j}{2 n \pi}) + \frac{H_1^2}{2E} \right] \\ - \sum_{m=1}^{\infty} a_m \pi^2 m^2 \left(\frac{\sin(m-n)\pi j}{(m-n)} - \frac{\sin(m+n)\pi j}{(m+n)} \right) (I_0 - I_2) \\ = \frac{t}{E n^3 \pi^3} \left[p(\cos n \pi k_1 - \cos n \pi k_2) - \beta w(1 - \cos n \pi) \right] \quad (35)_a \end{aligned}$$

Since the total number of coefficients is set at six, then when $n = 1$, m can only have the values of 3 and 5 and eq.(35)_a takes the form indicated in eq.(36).^{*} When $n = 3$, then $m = 1$ and 5 and eq.(37)^{*} is the result. When $n = 5$, $m = 1$ and 3 and this is eq.(38).^{*} When n is even, m must be even so that part is separate from the part when n is odd. This brings about three equations with three unknowns, a_1 , a_3 , and a_5 . These can be solved for by the usual method of elimination. The equations involving the even values of n and m are solved in the same way. With the values of a_1 to a_6 , inclusive the deflection is computed using eq.(20) and the bending moment by eq.(1). The bending moments are given in Table IV in the column headed Case A.

In the assumption for Case B the moment of inertia at the ends of the truss, I_1 , is taken as 16,440 in.²ft.² and I_0 , the moment of inertia over the center half is the same as previously, 47,150 in.²ft.² The moment of inertia between 0 and the quarter point is assumed to vary as a straight line from 16,440 in.²ft.² to 47,150 in.²ft.² Eq.(34) is changed to eq.(39) for the purposes of solution.

$$\begin{aligned} a_n \left[\frac{n^2 \pi^2}{2} (I_0 - (I_0 - I_1)(j - \frac{1 - \cos 2 n \pi j}{2 n^2 \pi^2 j}) + \frac{H_1^2}{2E} \right] \\ - \sum_{m=1}^{\infty} a_m (I_0 - I_1) \frac{m^2}{j} \left(\frac{1 - \cos(m-n)\pi j}{(m-n)^2} - \frac{1 - \cos(m+n)\pi j}{(m+n)^2} \right) \\ = \frac{t}{n^3 \pi^3 E} \left[p(\cos n \pi k_1 - \cos n \pi k_2) - \beta w(1 - \cos n \pi) \right] \quad (39) \end{aligned}$$

^{*}See next page.

$$\begin{aligned}
& a_1 \left[\pi^2 \left(\frac{I_0}{2} - (I_0 - I_2) \left(j - \frac{\sin 2\pi j}{2\pi} \right) + \frac{Hl^2}{2E} \right) - a_3 \, 9\pi \left(\frac{\sin 2\pi j}{2} - \frac{\sin 4\pi j}{4} \right) (I_0 - I_2) \right. \\
& \left. - a_5 \, 25\pi \left(\frac{\sin 4\pi j}{4} - \frac{\sin 6\pi j}{6} \right) (I_0 - I_2) = \frac{l}{E\pi} \left[p(\cos \pi k_1 - \cos \pi k_2) - 2\beta w \right] \right] \quad (36)
\end{aligned}$$

$$\begin{aligned}
& a_3 \left[9\pi^2 \left(\frac{I_0}{2} - (I_0 - I_2) \left(j - \frac{\sin 6\pi j}{6\pi} \right) + \frac{Hl^2}{2E} \right) - a_1 \pi \left(\frac{\sin - 2\pi j}{-2} - \frac{\sin 4\pi j}{4} \right) (I_0 - I_2) \right. \\
& \left. - a_5 \, 25\pi \left(\frac{\sin 2\pi j}{2} - \frac{\sin 8\pi j}{8} \right) (I_0 - I_2) = \frac{l}{27\pi E} \left[p(\cos 3\pi k_1 - \cos 3\pi k_2) - 2\beta w \right] \right] \quad (37)
\end{aligned}$$

$$\begin{aligned}
& a_5 \left[25\pi^2 \left(\frac{I_0}{2} - (I_0 - I_2) \left(j - \frac{\sin 10\pi j}{10\pi} \right) + \frac{Hl^2}{2E} \right) - a_1 \pi \left(\frac{\sin - 4\pi j}{-4} - \frac{\sin 6\pi j}{6} \right) (I_0 - I_2) \right. \\
& \left. - a_3 \, 9\pi \left(\frac{\sin - 2\pi j}{-2} - \frac{\sin 8\pi j}{8} \right) (I_0 - I_2) = \frac{l}{125\pi E} \left[p(\cos 5\pi k_1 - \cos 5\pi k_2) - 2\beta w \right] \right] \quad (38)
\end{aligned}$$

The solution for the coefficients a_1 and a_2 is exactly the same as the previous method. The resultant bending moments are indicated in Table IV in the column headed Case B. The values given by eq.(39) differ but slightly from those given by eq.(35) throughout the range of maximum moment from .2% to .3%. Of the two cases proposed, the second is preferred because the assumed curve of moment of inertia agrees more closely with the moment of inertia curve than does the assumed curve in Case A. Both of these methods presented give a shorter solution than the method of Turneaure.

In order to observe the accuracy of solution by the first method, an arbitrary change was made in I_1 and I_2 . They were assumed 50,000 in.² ft.² and 25,000 in.² ft.² respectively. H was considered to be unchanged. The bending moments were computed using both the method of Turneaure and Case A and the results tabulated in Table V. Another arbitrary change was made in I_1 and I_2 , this time to 60,000 in.² ft.² and 20,000 in.² ft.² respectively. These moments are also placed in Table V. In the first of these two arbitrary changes in I_1 and I_2 , the difference between the two methods through the range of maximum moments is about 2% which in the second is as much as 5%, indicating very good agreement between the two methods of solution. However, it is not expected to find I_1 and I_2 in the ratio of 3 to 1 or even as high as 2 to 1, so the method A of solution certainly presents as accurate a solution in less time than that of Turneaure.

Tables are presented in the Appendix for the evaluation of some of the quantities in eq.(39). In general, the term j will be located between the values of .2 and .3. The Tables are computed on this basis.

The deflections as determined by each method for $I_1 = 47,150$, $I_2 = 35,100$ and $I_3 = 16,440$ are compiled in Table IV as well as the deflections under constant moment of inertia.

Using case B, the Manhattan Bridge was studied for the live load which produced maximum moment at the center of the main span. Turneaure found this live load to extend from .3% to .7% and $H = 3,155,000$ lbs. The moment at the center as computed by case B is 92,050,000 ft. lbs., which is in agreement with that given by Turneaure of 91,250,000 ft. lbs.

An attempt was made to study the effect of the variable moment of inertia on the Ambassador bridge at Detroit, Michigan, but the variation of the moment of inertia in that bridge is so slight as to preclude the finding of any changes in the bending moments as far as any practical purposes are concerned.

TABLE IV

Bending Moments in foot-kips				Deflections in feet				
x	C.M.I.	Eq. 9	Case A	Case B	C.M.I.	Eq. 3	Case A	Case B
.14	79,970	76,053	76,240	75,756	4.2071	4.5204	4.5054	4.5441
.24	115,481	110,564	110,910	111,973	7.1540	7.5472	7.5196	7.4346
.254	119,117	115,419	115,506	117,060	7.9507	8.2464	8.2395	8.1152
.34	113,778	111,640	111,531	113,366	8.2611	8.4321	8.4409	8.2941
.44	74,739	74,201	74,444	75,125	7.5384	7.5814	7.5621	7.5075
.54	13,748	13,748	13,890	14,002	5.6384	5.6384	5.6272	5.6182
.64	-22,012	-22,758	-22,737	-22,511	3.4712	3.4509	3.4492	3.4311
.74	-40,678	-39,819	-39,927	-40,191	1.6566	1.5876	1.5966	1.6177
.84	-42,949	-41,080	-41,223	-41,509	.4906	.3411	.3526	.3756
.94	-30,257	-28,768	-28,540	-28,269	.0078	-.1113	-.1295	-.1512

Eq. 9

Case A

Case B

C.M.I.

I_a I_sI_a

Case A

I₁I_a

TABLE V

Bending Moments in ft. kips.							
x	C.M.I.	I _g = 25,000		I _c = 50,000		I _e = 60,000	
		Eq. 9	Case A	Eq. 9	Case A	Eq. 9	Case A
.1 <i>l</i>	79,970	68,596	69,540	63,775	66,594	63,775	66,594
.2 <i>l</i>	115,481	100,648	102,844	94,899	100,759	94,899	100,759
.25 <i>l</i>	119,117	107,084	109,074	103,159	108,735	103,159	108,735
.3 <i>l</i>	113,778	105,665	107,108	104,004	108,436	104,004	108,436
.4 <i>l</i>	74,739	71,256	72,256	71,767	74,662	71,767	74,662
.5 <i>l</i>	13,748	12,525	12,879	13,214	14,626	13,214	14,626
.6 <i>l</i>	-22,012	-22,675	-22,447	-22,158	-21,307	-22,158	-21,307
.7 <i>l</i>	-40,678	-38,171	-37,800	-37,104	-36,267	-37,104	-36,267
.8 <i>l</i>	-42,949	-37,885	-37,899	-35,299	-35,545	-35,299	-35,545
.9 <i>l</i>	-30,257	-26,100	-26,227	-24,187	-24,325	-24,187	-24,325

CONCLUSIONS

1. The horizontal component of the tension in the cable is, for all practical purposes, independent of the moment of inertia of the stiffening truss. With a 40% change in the average value of the moment of inertia for a long span bridge, H_p changes less than 1%.

2. The bending moment and deflection are not independent of the moment of inertia. However, if I_a does not change more than 10%, the bending moment and deflection may each be considered as being directly proportional to the average value of the moment of inertia.

3. The formulas for deflection and bending moment, when a variable moment of inertia is used, are developed easily when a trigonometric series is used to represent the deflection. The coefficients a_1 to a_n are determined with only slightly more trouble than when the moment of inertia is constant. The general form of a_n indicates its ready adaptation to a change in the condition of loading or to a change in the values of the moments of inertia used.

a. The average value of the moment of inertia may be used when I_{max} is not more than 50% larger than I_{min} .

b. When there is considerable variation in the moment of inertia, such as the type indicated in Fig. 1 for the Manhattan and Philadelphia-Camden Bridges, it is desirable to use the assumption of Case B so as to obtain a more exact solution than that given by using the assumption of an average moment of inertia. A similar accuracy could be obtained, using the method of Turneaure by dividing the moment of inertia curve into a large number of sections, but that method would involve much more work in the solution than the method presented in Case B.

APPENDIX

PREVIOUS METHODS

In 1930, E. Steuermann⁷ presented a method for the solution for the bending moment in the stiffening truss when the moment of inertia is not considered as constant. The usual moment equation is transformed into the following form noting that $\beta = \frac{H_2}{H_1}$.

$$EI \frac{d^2 \eta}{dx^2} - H_w(1 + \beta)\eta = -M_1 + \beta H_w y \quad (36)$$

Let I_1 be the moment of inertia at the end of the beam and I_c the moment of inertia at the center with $I_c > I_1$, which is a reasonable assumption if the truss is not continuous at the towers. The value of I at any point may be taken as $I = I_c \sin(\frac{m\pi x}{l} + \alpha)$ where $\alpha = \sin^{-1} \frac{I_1}{I_c}$ and $m = 1 - \frac{2\alpha}{\pi}$. This is a definite restriction on the value of I and is "a special case of $I = I_c f(x)$."

After expanding the right member of eq.(36) in a Fourier Series,

$$-M + \beta H_w y = -(b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \dots + b_n \sin \frac{n\pi x}{l}) \quad (37)$$

Where n may have any positive integral value, and where

$$b_n = \frac{2}{l} \int_0^l (M - \beta H_w y) \sin \frac{n\pi x}{l} dx \quad (38)$$

and expressing η by a series,

$$\eta = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \dots + a_n \sin \frac{n\pi x}{l} \quad (39)$$

and substituting in eq.(1), the following equation results,

$$\frac{\pi^2 EI}{l^2} \sin(\frac{m\pi x}{l} + \alpha) \sum_1^{\infty} n^2 a_n \sin \frac{n\pi x}{l} + H \sum_1^{\infty} a_n \sin \frac{n\pi x}{l} = \sum_1^{\infty} b_n \sin \frac{n\pi x}{l} \quad (40)$$

To find the coefficient a_n , multiply both sides of eq.(40) by $\sin \frac{k\pi x}{l}$ and integrals between the limits of 0 and l , where $k = 1, 2, 3, \dots, n$. After integration and transposition, the following equation results:

$$a_k + \frac{4\pi EI_c}{l} \frac{m \cos \alpha}{H} \sum \frac{[(-1)^{n+k+1} - 1]}{[(n-k)^2 - m^2]} \frac{k n^2 a_n}{[(n+k)^2 - m^2]} = \frac{b_k}{H} \quad (41)$$

⁷ibid

or

$$a_k + \mu \sum_{n=1}^{\infty} N_{kn} a_n = C_k \quad (42)$$

In eq.(41) it will be noted that n and k must both be odd or both be even, otherwise the second term will become zero. As a result of this, the solution is divided very neatly in two parts, one for even values of k and one for odd values.

The author considered it necessary to carry the expansion of μ to only four terms, therefore it is only necessary to set up two pair of simultaneous equations and make the solution for a_1 , a and a_2 , and a . The deflection may be easily found by eq.(39) and then the bending moment by eq.(36).

The advantage of this method is that it is a more exact solution since the assumed moment of inertia curve is made to very closely approximate the designed curve. This method produces more satisfactory results near the end than the method of Turneaure. A great difficulty presents itself in the assumption of a correct value of $f(x)$ in the equation $I = I f(x)$. The correct value of $f(x)$ may cause considerable difficulty in the integration, or it may even be impossible to integrate. As to the number of terms to be used in the equation for η , A. A. Jakkula considers at least five to be necessary.

In 1928, G. B. Martin presented a solution for the bending moment in a stiffening truss with a variable moment of inertia. This was not a general solution of the problem but a solution for load over the half span only. The assumptions made, are applicable to half span loading only, so a general solution cannot be evolved from this particular solution. Inasmuch as this is a very particular solution, it will not be reviewed here, but only mentioned that the solution does exist.

TABLE VI
Values of $(\cos n\pi k_1 - \cos n\pi k_2)$

k	$\cos \pi k$	$\cos 2\pi k$	$\cos 3\pi k$	$\cos 4\pi k$	$\cos 5\pi k$	$\cos 6\pi k$
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.025	.99692	.98769	.97237	.95106	.92388	.89101
.050	.98769	.95106	.89101	.80902	.70711	.58779
.075	.97237	.89101	.76041	.58779	.38268	.15643
.100	.95106	.80902	.58779	.30902	0	-.30902
.125	.92388	.70711	.38268	0	-.38268	-.70711
.150	.89101	.58779	.15643	-.30902	-.70711	-.95106
.175	.85264	.45399	-.07846	-.58779	-.92388	-.98769
.200	.80902	.30902	-.30902	-.80902	-1.00000	-.80902
.225	.76041	.15643	-.52250	-.95106	-.92388	-.45399
.250	.70711	0	-.70711	-1.00000	-.70711	0
.275	.64945	-.15643	-.85264	-.95106	-.38268	.45399
.300	.58779	-.30902	-.95106	-.80902	0	.80902
.325	.52250	-.45399	-.99692	-.58779	.38268	.98769
.333	.50000	-.50000	-1.00000	-.50000	.50000	1.00000
.350	.45399	-.58779	-.98769	-.30902	.70711	.95106
.375	.38268	-.70711	-.92388	0	.92388	.70711
.400	.30902	-.80902	-.80902	.30902	1.00000	.30902
.425	.23345	-.89101	-.64945	.58779	.92388	-.15643
.450	.15643	-.95106	-.45399	.80902	.70711	-.58779
.475	.07846	-.98769	-.23345	.95106	.38268	-.89101

TABLE VI (Continued)

k	$\cos \pi k$	$\cos 2\pi k$	$\cos 3\pi k$	$\cos 4\pi k$	$\cos 5\pi k$	$\cos 6\pi k$
.500	0	-1.00000	0	1.00000	0	-1.00000
.525	-.07846	-.98769	.23345	.95106	-.38268	-.89101
.550	-.15643	-.95106	.45399	.80902	-.70711	-.58779
.575	-.23345	-.89101	.64945	.58779	-.92388	-.15643
.600	-.30902	-.80902	.80902	.30902	-1.00000	.30902
.625	-.38628	-.70711	.92388	0	-.92388	.70711
.650	-.45399	-.58779	.98769	-.30902	-.70711	.95106
.667	-.50000	-.50000	1.00000	-.50000	-.50000	1.00000
.675	-.52250	-.45399	.99692	-.58779	-.38268	.98769
.700	-.58779	-.30902	.95106	-.80902	0	.80902
.725	-.64945	-.15643	.85264	-.95106	.38268	.45399
.750	-.70711	0	.70711	-1.00000	.70711	0
.775	-.76041	.15643	.52250	-.95106	.92388	-.45399
.800	-.80902	.30902	.30902	-.80902	1.00000	-.80902
.825	-.85264	.45399	.07846	-.58779	.92388	-.98769
.850	-.89101	.58779	-.15643	-.30902	.70711	-.95106
.875	-.92388	.70711	-.38628	0	.38268	-.70711
.900	-.95106	.80902	-.58779	.30902	0	-.30902
.925	-.97237	.89101	-.76041	.58779	-.38268	.15643
.950	-.98769	.95106	-.89101	.80902	-.70711	.58779
.975	-.99692	.98769	-.97237	.95106	-.92388	.89101
1.000	-1.00000	1.00000	-1.00000	1.00000	-1.00000	1.00000

TABLE VII

Values of $(j - \frac{1 - \cos 2 \pi n j}{2 n^2 n^2 j})$ when $.20 < j < .30$

$j \backslash n$	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30
1	.024973	.028753	.032874	.037342	.042168	.047357	.052917	.058852	.065167	.071864	.078948
2	.085442	.096839	.108905	.121598	.134873	.148679	.162960	.177658	.192711	.208056	.223628
3	.149086	.164846	.180704	.196517	.212151	.227484	.242407	.256827	.270669	.283877	.296417
4	.189061	.203001	.216099	.228297	.239585	.250000	.259617	.268550	.276935	.284932	.292707
5	.200000	.209528	.218241	.226368	.234166	.241894	.249798	.258053	.266908	.276367	.286491
6	.195138	.202878	.210880	.219421	.228685	.238742	.249555	.260989	.272834	.284843	.296759

TABLE VIII

Values of $\frac{m^2}{j} \left[\frac{1 - \cos(m-n)\pi}{(m-n)^2} - \frac{1 - \cos(m+n)\pi}{(m+n)^2} \right]$ $.20 < j < .30$

$\frac{j}{m}$	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30
3 1	2.685656	3.023920	3.376790	3.742092	4.117336	4.500000	4.887368	5.276542	5.664656	6.048718	6.425738
5 1	7.851649	8.390038	8.857876	9.242708	9.533760	9.722222	9.801355	9.767040	9.617646	9.354041	8.979895
1 3	.298406	.335991	.375199	.415788	.457482	.500000	.543041	.586282	.629406	.672080	.713971
5 3	20.243555	21.497005	22.604563	23.558135	24.355371	25.000000	25.500631	25.870356	26.125907	26.286194	26.371536
1 5	.314066	.335602	.354315	.369708	.381350	.388889	.392054	.390652	.384706	.374162	.359196
3 5	7.287680	7.738922	8.137643	8.480929	8.767934	9.000000	9.180227	9.313328	9.405326	9.463030	9.493753
4 2	9.799556	10.745482	11.672222	12.567942	13.421315	14.222222	14.961521	15.631424	16.225794	16.739862	17.170667
6 2	18.408094	18.860009	19.043412	18.955345	18.602390	18.000000	17.171437	16.147146	14.962681	13.657249	12.272062
2 4	2.449889	2.686370	2.918056	3.141986	3.355329	3.555556	3.740380	3.907856	4.056448	4.184966	4.292667
6 4	31.094100	32.115103	32.931033	33.581019	34.108905	34.560000	34.976395	35.393946	35.838746	36.330443	36.870600
2 6	2.045344	2.095556	2.115935	2.106149	2.066932	2.000000	1.907937	1.794127	1.662520	1.517472	1.363562
4 6	13.819600	14.273379	14.636015	14.924897	15.159513	15.360000	15.545064	15.730643	15.928332	16.146864	16.386933

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